1. $(f,g) = \int_0^\infty \exp(-3x)\, f(x)\, g(x)\, dx$

a. $\varphi_0(x) = 1$

$\varphi_1(x) = x - \alpha \qquad (\varphi_0, \varphi_1) = 0$

$$(\varphi_0, \varphi_1) = \int_0^\infty \exp(-3x)(x-\alpha)\, dx$$

$$= -\alpha \int_0^\infty \exp(-3x)\, dx + \int_0^\infty x \exp(-3x)\, dx$$

$$= \frac{\alpha}{3} \exp(-3x) \Big|_0^\infty + \Big[-\frac{1}{3} x \exp(-3x) \Big|_0^\infty - -\frac{1}{3} \int_0^\infty \exp(-3x)\, dx \Big]$$

($=0$)

$$= -\frac{\alpha}{3} - \frac{1}{9} \exp(-3x) \Big|_0^\infty = -\frac{\alpha}{3} + \frac{1}{9}$$

$\Rightarrow -\frac{\alpha}{3} + \frac{1}{9} = 0 \quad \Rightarrow \alpha = \frac{1}{3}$

$\Rightarrow \varphi_1(x) = x - \frac{1}{3}$

b. • We can approximate $\int_0^\infty \exp(-3x) f(x)\, dx$ by Gaussian rule

0.4

• 1. zero of $\varphi_1(x) = x - \frac{1}{3}$ is $x_0 = \frac{1}{3}$

0.3

2. Gauss rule with 1 interpolation point x_0:

exact integration of $f(x_0)$

$$\Rightarrow \int_0^\infty \exp(-3x)\, f(x_0)\, dx = f(x_0) \int_0^\infty \exp(-3x)\, dx$$

0.8

$$= \frac{1}{3} f(x_0) \qquad \text{with } x_0 = \frac{1}{3}$$

3. Gauss rule: $\int_0^\infty \exp(-3x)\, f(x_0)\, dx = \frac{1}{3} f\left(\frac{1}{3}\right)$

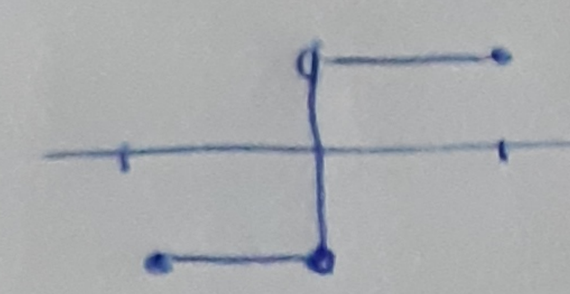
c. Gauss is exact for polynomials of order $2n-1$ where n is the number of interpolation points

polynomial of degree n has n zero's $\Rightarrow n$ interpolation points

(now: $n=1 \Rightarrow$ Gauss exact for 1st order polynomial)

Exc 2

$$f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in (0, 1) \end{cases}$$



a. $C_n(x) = \sum_{k=0}^{n} a_k T_k(x)$

0.4

- $a_k = \dfrac{(f(x), T_k(x))}{(T_k(x), T_k(x))}$
- T_k even for k even
  T_k odd for k odd

0.2

$$(f(x), T_k(x)) = \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{even}} \underbrace{f(x)}_{\text{odd}} T_k(x)\, dx$$

- integration interval symmetric around 0
  $\Rightarrow$ integral 0 if integrand is odd

0.2

- $\frac{1}{\sqrt{1-x^2}} f(x) T_k(x)$ is odd if T_k even, k even
  since then integrand = even x odd x even = odd

0.2

$\Rightarrow (f(x), T_k(x)) = 0$ for k even

$\Rightarrow a_k = 0$ for k even

b. show $a_k = \frac{4}{k\pi} (-1)^{\frac{k-1}{2}}$ for k odd

k odd

0.7

$$(f(x), T_k(x)) = \int_0^1 \frac{1}{\sqrt{1-x^2}} T_k(x)\, dx - \int_0^1 \frac{1}{\sqrt{1-x^2}} T_k(x)\, dx$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$T_k(\cos\theta) = \cos(k\theta)$

$$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}} T_k(x)\, dx = 2\int_{\frac{\pi}{2}}^{0} \frac{1}{|\sin\theta|} \cos(k\theta) . -\sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin\theta}{|\sin\theta|} \cos k\theta\, d\theta = 2\int_0^{\pi/2} \cos k\theta\, d\theta$$

$k=1: +1$ $\quad k=3: -1$ $\quad k=5: +1$ etc

$$= \frac{2}{k} \sin k\theta \Big|_0^{\pi/2} = \frac{2}{k} \sin\frac{k\pi}{2} = \frac{2}{k} (-1)^{\frac{k-1}{2}} \quad k \text{ odd}$$

$$(T_k(x), T_k(x)) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_k(x) T_k(x)\, dx$$

$x = \cos\theta$

$$= \int_{\pi}^{0} \frac{1}{|\sin\theta|} \cos(k\theta)\cos(k\theta) \cdot -\sin\theta \, d\theta$$

0.7

$$= \int_{0}^{\pi} \frac{\sin\theta}{|\sin\theta|} \cos^2(k\theta)\, d\theta = \int_{0}^{\pi} \cos^2(k\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (\cos 2k\theta + 1)\, d\theta = \frac{1}{2}\left(\frac{1}{2k}\sin 2k\theta + \theta\right)\Big|_0^{\pi} = \frac{1}{2}\pi$$

$\cos 2\theta = 2\cos^2\theta - 1$

0.2 $\Rightarrow$

$$a_k = \frac{(f, T_k)}{(T_k, T_k)} = \frac{\frac{2}{k}(-1)^{\frac{k-1}{2}}}{\frac{1}{2}\pi} = \frac{4}{k\pi}(-1)^{\frac{k-1}{2}}$$

0.2 $C_3 = a_1 T_1 + a_3 T_3$ since $a_0 = a_2 = 0$

$T_{n+1} = 2x\, T_n - T_{n-1}$ $\Rightarrow$ $\begin{cases} T_0 = 1 \\ T_1 = x \end{cases}$ start

(*)

0.4

$T_2 = 2x\, x - 1 = 2x^2 - 1$

$T_3 = 2x(2x^2 - 1) - x = 4x^3 - 3x$

$$\Rightarrow C_3 = \frac{4}{\pi} T_1 + \frac{4}{3\pi} \cdot -1 \cdot T_3$$

0.3

$$= \frac{4}{\pi} x - \frac{4}{3\pi}(4x^3 - 3x) = -\frac{16}{3\pi} x^3 + \left(\frac{4}{\pi} + \frac{4}{\pi}\right) x$$

$$C_3 = -\frac{16}{3\pi} x^3 + \frac{8}{\pi} x$$

(*) or: $T_3(\cos\theta) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$x = \cos\theta$

$\Rightarrow T_3(x) = 4x^3 - 3x$

c. • $C_4(x) = C_3(x) = a_1 T_1(x) + a_3 T_3(x)$
↑ because $a_4 = 0$ (see part (a))

$C_3(x)$ is odd (T_1, T_3 odd)

0.2 $\Rightarrow C_4(x)$ odd
$f(x)$ is odd
} $\Rightarrow$ error $= C_4(x) - f(x) \Rightarrow$ odd error
(↑ odd, ↑ odd)

• since C_4 is polynomial of order $n=4$
Vallee Poussin applicable in case of $n+2$ extrema:
let: $n+2$ points $x_0 < x_1 < \dots < x_{n+1}$.
if $\exists g_n$ error $f(x_j) - g_n(x_j) = (-1)^j e_j$ $j = 0 \dots n+1$
g_n degree polynomal $\leq n$
then $\min |e_j| \leq E_n^*(f)$
where $E_n^*(f)$ is minimax norm

now $n=4$, in error plot we see 6 extrema (2 at $x=0$)
hence we can apply la Vallee Poussin
↑ caused by discontinuity of f

hence Vallee Poussin gives
$\min |e_j| \leq E_n^*(f)$ smallest abs. error
from plot we see at $x \approx \pm 1$ $\Rightarrow |e_j| \approx 0.1$

0.3 $\Rightarrow 0.1 \leq E_n^*(f)$

Furthermore $E_n^*(f) \leq \max |e_j| = 1$ (at $x=0$)
$\Rightarrow 0.1 \leq E_n^*(f) \leq 1$

d. • We will have convergence in the norm associated to the
0.2 innerproduct that defines the Cheb. polynomials (L_2-norm)
0.3 • We will not have pointwise convergence since f not continuous